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A multilinear form inequality

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Abstract

An inequality of Interpolation type for Multilinear Forms with a two-part dependence condition is proved. It generalizes the work of Bradley and Bryc [Theorem 3.6, Multilinear forms and measures of dependence between random variables, *J. Multivariate Anal*. 16 (1985) 335–367] and Prakasa Rao [Bounds for rth order joint cumulant under rth order strong mixing, *Statist. Probab. Lett.* 43 (1999) 427–431].

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1. Introduction

We start with some definitions and notations. If W is nonnegative random variable on a probability space $(\Omega, \mathcal{F}, P)$, then for each $z \in (0, 1)$, define the quantile

$$Q_W(z) := \inf\{t \geqslant 0 : P(W > t) \leqslant z\}.$$

We notice that $t \geqslant Q_W(z)$ if and only if $P(W > t) \leqslant z$.

For any set A, I_A or $I(A)$ will denote the indicator function of A. Let $\mathcal{A}$ be any σ-field $\subset \mathcal{F}$. A simple random variable is a random variable for which the range is a finite set. We denote by $\mathcal{S}(\mathcal{A})$ the set of $\mathbb{C}$-valued $\mathcal{A}$-measurable simple random variables. If $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are σ-fields, a function $F : \mathcal{S}(\mathcal{A}_1) \times \cdots \times \mathcal{S}(\mathcal{A}_r) \to \mathbb{C}$ is said to be a multilinear form if it is linear in each of

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its variables. If in addition for all $X_1 \in \mathcal{S}(\mathcal{A}_1), \ldots, X_r \in \mathcal{S}(\mathcal{A}_r)$ we have that

$$|F(X_1, \ldots, X_r)| \leqslant \|X_k\|_1 \cdot \prod_{i \neq k} \|X_i\|_\infty \quad \text{for } k = 1, \ldots, r \tag{1}$$

then we say that F is a multilinear product form. Multilinear forms arise in the study of dependence coefficients. A very common example of a multilinear form is $Cov(X_1, X_2)$. More generally when $r \geqslant 2$, there is the multivariate cumulant:

$$Cum(X_1, \ldots, X_r) := \sum (-1)^{p-1}(p-1)! \left(E \prod_{j \in v_1} X_j \right) \ldots \left(E \prod_{j \in v_p} X_j \right),$$

where the sum is taken over the partitions $(v_1, \ldots, v_p)$ of $\{1, \ldots, r\}$ for $p = 1, \ldots, r$. If one multiplies the cumulant by $(\sum_{p=1}^r p^r \cdot (p-1)!)^{-1}$, then it satisfies (1) and thus it becomes a product form. In [3], some inequalities concerning multilinear product forms are proved using interpolation theory. The main goal of this paper is to prove the following theorem:

Theorem 1. *Suppose r is a positive integer and that $\mathbf{p} = (p_1, \ldots, p_r) \in [1, \infty]^r$ is such that $\sum_{i=1}^r p_i^{-1} \leqslant 1$. Define the number $c = Card\{i \in \{1, \ldots, r\} : p_i < \infty\} - \sum_{i=1}^r p_i^{-1}$. Then there exists a constant $C = C(\mathbf{p})$ that is a function only of $\mathbf{p}$ such that the following holds: suppose $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are σ-subfields of $\mathcal{F}$, $F : \mathcal{S}(\mathcal{A}_1) \times \cdots \times \mathcal{S}(\mathcal{A}_r) \to \mathbb{C}$ is a multilinear product form and α and λ are numbers in $[0, 1]$ such that for every choice of events $A_i \in \mathcal{A}_i$, one has that*

$$|F(I_{A_1}, \ldots, I_{A_r})| \leqslant \alpha + \lambda \cdot \prod_{i=1}^r [P(A_i)]^{1/p_i}; \tag{2}$$

then for all $X_1 \in \mathcal{S}(\mathcal{A}_1), \ldots, X_r \in \mathcal{S}(\mathcal{A}_r)$, we have

$$|F(X_1, \ldots, X_r)| \leqslant 4^r \cdot \int_0^\alpha \prod_{i=1}^r \mathcal{Q}_{|X_i|}(z)\, dz + C \cdot \lambda(1 - \log(\lambda))^c \cdot \prod_{i=1}^r \|X_i\|_{p_i}.$$

The case when $\alpha = 0$ was proved in [3]. Rio proved several results related to the case $\lambda = 0$ (see for example [8, Theorem 1.1]). In [6] Prakasa Rao studied the case in which F is the cumulant function and $\sum_{i=1}^r p_i^{-1} = 1$. By a simple application of Riesz–Thorin's Multilinear Interpolation Theorem (see [1, p. 18, Exercise 13]), if F is a multilinear product form and $\mathbf{p} = (p_1, \ldots, p_r) \in [1, \infty]^r$ is such that $\sum_{i=1}^r p_i^{-1} \leqslant 1$, then for all $X_1 \in \mathcal{S}(\mathcal{A}_1), \ldots, X_r \in \mathcal{S}(\mathcal{A}_r)$, we have $F(X_1, \ldots, X_r) \leqslant \prod_{i=1}^r \|X_i\|_{p_i}$ (see [3, p. 349] for details). However, Theorem 1 gives a bound for F that gets small as λ and α get small.

The assumption (2) is called a two-part dependence condition. Peligrad [5] proved a CLT for a class of strictly stationary sequences that satisfy a two-part dependence condition. These sequences arise, for example, when one applies a certain nonlinear smoothing algorithm of Tukey [9] (the "3R" or "running median" smoother) to strictly stationary ρ-mixing sequences (see [4]).

The CLT in [5] involved the parameters λ and α both getting small. In such problem as finding a CLT for the estimators of cumulants (themselves or their spectral densities), we would like the parameters λ and α to be small so that Theorem 1 could be applied.

2. Preliminaries

We will use the following lemmas.

Lemma 1. *Suppose that the simple random variables* $X_1 \in \mathcal{S}(\mathcal{A}_1), \ldots, X_r \in \mathcal{S}(\mathcal{A}_r)$ *are nonnegative real-valued. Then*

$$F(X_1, \ldots, X_r) = \int_0^\infty \cdots \int_0^\infty F(I(X_1 > t_1), \ldots, I(X_r > t_r))\, dt_1 \ldots dt_r.$$

Proof. Let $0 = b_{i,0} < b_{i,1} < b_{i,2} < \cdots < b_{i,n_i}$ the values in the range of X_i, $i = 1, \ldots, r$ with the possible exception of $b_{i,0} = 0$. Then

$$X_i = \sum_{j=0}^{n_i} b_{i,j} \cdot I(X_i = b_{i,j}) = \sum_{j=1}^{n_i} (b_{i,j} - b_{i,j-1}) I(X_i \geqslant b_{i,j}).$$

Using the multilinearity of F we get

$$\begin{aligned} &F(X_1, \ldots, X_r) \\ &= \sum_{j_1=1}^{n_1} \cdots \sum_{j_r=1}^{n_r} \prod_{i=1}^{r} (b_{i,j_i} - b_{i,(j_i-1)}) F(I(X_1 \geqslant b_{1,j_1}), \ldots, I(X_r \geqslant b_{r,j_r})) \\ &= \sum_{j_1=1}^{n_1} \cdots \sum_{j_r=1}^{n_r} \int_{b_{1,(j_1-1)}}^{b_{1,j_1}} \ldots \int_{b_{r,(j_r-1)}}^{b_{r,j_r}} F(I(X_1 > t_1), \ldots, I(X_r > t_r))\, dt_1 \ldots dt_r \\ &= \int_0^\infty \ldots \int_0^\infty F(I(X_1 > t_1), \ldots, I(X_r > t_r))\, dt_1 \ldots dt_r. \qquad \square \end{aligned}$$

The following result is essentially due to Rio [7].

Lemma 2. *Suppose that* $X_1, \ldots, X_r$ *are nonnegative real-valued simple random variables. Then*

$$\int_0^\infty \ldots \int_0^\infty \min\{P(X_1 > t_1), \ldots, P(X_r > t_r), \alpha\}\, dt_1 \ldots dt_r = \int_0^\alpha \prod_{i=1}^{r} Q_{X_i}(z)\, dz.$$

Proof. A proof is given by Prakasa Rao [6, Eq. (1.1)]. $\square$

Lemma 3. *Suppose* L *and* M *are each a positive integer, and* $t > 0$. *Then*

$$\sum_{n_1=L}^{\infty} \cdots \sum_{n_M=L}^{\infty} 2^{-t \max\{n_1, \ldots, n_M\}} \leqslant M!\, \frac{2^{-t \cdot L}}{(1 - 2^{-t})^M}.$$

Proof. We notice that

$$\sum_{n_1=L}^{\infty} \cdots \sum_{n_M=L}^{\infty} 2^{-t\ \max\{n_1,\ldots,n_M\}}$$
$$\leqslant M! \sum_{n_1=L}^{\infty} \sum_{n_2=n_1}^{\infty} \cdots \sum_{n_{M-1}=n_{M-2}}^{\infty} \sum_{n_M=n_{M-1}}^{\infty} 2^{-t\cdot n_M}$$

and the result follows by applying M times the geometric series formula. □

We recall the fact that any complex random variable Y can be written as $Y = Y_1 - Y_2 + \sqrt{-1} \cdot (Y_3 - Y_4)$, where Y_j for $j = 1, \ldots, 4$ are nonnegative random variables with $Y_j \leqslant |Y|$. First recall that any complex random variable can be written in the form $Y = \Re(Y) + \sqrt{-1} \cdot \Im(Y)$, where $\Re(Y)$ and $\Im(Y)$ denote respectively the real and imaginary part of Y. Then the random variables $Y_1 = \max\{\Re(Y), 0\}$, $Y_2 = \max\{-\Re(Y), 0\}$, $Y_3 = \max\{\Im(Y), 0\}$ and $Y_4 = \max\{-\Im(Y), 0\}$ will have the desired property. This, the multilinearity of F and the facts that $\mathcal{Q}_{Y_j} \leqslant \mathcal{Q}_{|Y|}$, $j = 1, \ldots, 4$ show that in order to prove Theorem 1, it is enough to prove the following proposition.

Proposition 1. *Suppose the hypothesis of Theorem* 1 *holds, and in addition the random variables* $X_1, \ldots, X_r$ *are real-valued and nonnegative. Then*

$$|F(X_1, \ldots, X_r)| \leqslant \int_0^{\alpha} \prod_{i=1}^{r} \mathcal{Q}_{X_i}(z)\, dz + \frac{C}{4^r} \cdot \lambda(1 - \log(\lambda))^c \prod_{i=1}^{r} \|X_i\|_{p_i}.$$

It follows from Proposition 1 that Theorem 1 remains valid with its final inequality replaced by the one above if we have the extra assumption that the random variables $X_1, \ldots, X_r$ are real and nonnegative.

The proof of Proposition 1 begins as follows:

Since F is a product form, for each $k \in \{1, \ldots, r\}$, we have

$$\begin{aligned} &|F(I(X_1 > t_1), \ldots, I(X_r > t_r))| \\ &\quad \leqslant \|I(X_k > t_k)\|_1 \prod_{i \neq k} \|I(X_i > t_i)\|_{\infty} \\ &\quad \leqslant \|I(X_k > t_k)\|_1 \\ &\quad = P(X_k > t_k). \end{aligned} \tag{3}$$

Using the main assumption (2),

$$|F(I(X_1 > t_1), \ldots, I(X_r > t_r))| \leqslant \alpha + \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i}. \tag{4}$$

Combining (3) and (4) we obtain

$$\begin{aligned} &|F(I(X_1 > t_1), \ldots, I(X_r > t_r))| \\ &\quad \leqslant \min\left\{ P(X_1 > t_1), \ldots, P(X_r > t_r), \alpha + \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i} \right\} \end{aligned}$$

$$\leqslant \min\{P(X_1 > t_1), \dots, P(X_r > t_r), \alpha\}$$
$$+ \min\left\{P(X_1 > t_1), \dots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i}\right\}.$$

Using this inequality and Lemma 1 we obtain

$$\begin{aligned}
&|F(X_1, \dots, X_r)| \\
&= \left| \int_0^\infty \dots \int_0^\infty F(I(X_1 > t_1), \dots, I(X_r > t_r))\, dt_1 \dots dt_r \right| \\
&\leqslant \int_0^\infty \dots \int_0^\infty |F(I(X_1 > t_1), \dots, I(X_r > t_r))|\, dt_1 \dots dt_r \\
&\leqslant \int_0^\infty \dots \int_0^\infty \min\{P(X_1 > t_1), \dots, P(X_r > t_r), \alpha\}\, dt_1 \dots dt_r \\
&\quad + \int_0^\infty \cdots \int_0^\infty \min\left\{ P(X_1 > t_1), \dots, P(X_r > t_r), \right. \\
&\quad \left. \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i} \right\} dt_1 \dots dt_r
\end{aligned}$$

and by Lemma 2 we know that

$$\int_0^\infty \dots \int_0^\infty \min\{P(X_1 > t_1), \dots, P(X_r > t_r), \alpha\}\, dt_1 \dots dt_r = \int_0^\alpha \prod_{i=1}^{r} \mathcal{Q}_{X_i}(z)\, dz.$$

So it is enough to prove the following:

$$\begin{aligned}
&\int_0^\infty \dots \int_0^\infty \min\left\{P(X_1 > t_1), \dots, P(X_r > t_r), \lambda \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i}\right\} dt_1 \dots dt_r \\
&\leqslant \frac{C}{4^r} \cdot \lambda(1 - \log(\lambda))^c \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}.
\end{aligned}$$

So we see that the proof of Proposition 1 reduces to the following lemma:

Lemma 4. *Suppose r is a positive integer and $\mathbf{p} = (p_1, \dots, p_r) \in [1, \infty]^r$ with $\sum_{i=1}^{r} p_i^{-1} \leqslant 1$, $\lambda \in [0, 1]$. Define the number $c = c(\mathbf{p}) := Card\{k : p_k < \infty\} - \sum_{i=1}^{r} p_i^{-1}$. Then there exists a constant $C = C(\mathbf{p})$ that is a function only of $\mathbf{p}$ such that the following holds*:

If $X_1, \dots, X_r$ are nonnegative simple random variables, then we have

$$\begin{aligned}
&\int_0^\infty \dots \int_0^\infty \min\left\{P(X_1 > t_1), \dots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i}\right\} dt_1 \dots dt_r \\
&\leqslant C \cdot \lambda(1 - \log(\lambda))^c \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}.
\end{aligned}$$

(Notice that C is basically the same as in Proposition 1 and Theorem 1 with the difference that it has been multiplied by $(4^r)^{-1}$.) Here and for the rest of the paper, log will denote the logarithm to the base e. As usual, $\log_2$ denotes the logarithm to the base 2.

3. Proof of Lemma 4

If $\lambda = 0$, the proof is trivial. So we can assume $0 < \lambda \leqslant 1$. We will separate the proof into four cases.

Case 1: $\mathbf{p} \in (1, \infty)^r$ and $\sum_{i=1}^r p_i^{-1} < 1$.

The proof will be written out here under the extra assumption that $r \geqslant 2$. If instead $r = 1$, the proof is similar but much simpler. For each $j \in \{1, 2, \dots, r\}$, define the positive constant $K_j = K_j(\mathbf{p})$ (where $\mathbf{p} = (p_1, \dots, p_r)$) as follows:

$$\begin{aligned} K_j &= K_j(\mathbf{p}) \\ &= 2\left(\prod_{k=1}^r \frac{1}{2^{1/p_k} - 1}\right) \\ &\quad \times \sum \left[(r - (1 + Card\ A))! \cdot \frac{1}{\left(1 - 2^{-(1/p_j + 1 - \sum_{k=1}^r 1/p_k)}\right)^{r-(1+Card\ A)}} \right], \end{aligned} \tag{5}$$

where the sum is taken over all sets $A \subsetneq \{2, 3, \dots, r\}$ (including the empty set). Note that it can be rewritten

$$\begin{aligned} K_j &= K_j(\mathbf{p}) \\ &= 2\left(\prod_{k=1}^r \frac{1}{2^{1/p_k} - 1}\right) \\ &\quad \times \sum_{l=0}^{r-2} \binom{r-1}{l} \left[(r - (1 + l))! \cdot \frac{1}{\left(1 - 2^{-(1/p_j + 1 - \sum_{k=1}^r 1/p_k)}\right)^{r-1-l}} \right]. \end{aligned}$$

For each $j \in \{1, \dots, r\}$, define the positive constant $C_j = C_j(\mathbf{p})$ as follows:

$$\begin{aligned} C_j &= C_j(\mathbf{p}) \\ &= 2^r K_j + 2^{r+2} \cdot [\max\{p_1, p_2, \dots, p_r\}]^{r-1}. \end{aligned}$$

Define the positive $C = C(\mathbf{p})$ as follows:

$$C = \sum_{j=1}^r C_j.$$

Now suppose $X_1, \dots, X_r$ are nonnegative simple random variables. Let $H : [0, \infty)^r \longrightarrow \mathbb{R}$ be defined as

$$H(t_1, \dots, t_r) := \min\left\{ P(X_1 > t_1), \dots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^r [P(X_i > t_i)]^{1/p_i} \right\}. \tag{6}$$

Our aim is to show that

$$\int_0^\infty \dots \int_0^\infty H(t_1, \dots, t_r)\, dt_1 \dots dt_r \leqslant C \cdot \lambda (1 - \log(\lambda))^c \cdot \prod_{i=1}^r \|X_i\|_{p_i}.$$

For each $i = 1, \ldots, r$, define the nondecreasing sequence $a_{i,n}$ by

$$a_{i,0} = 0,$$
$$a_{i,n} = \mathcal{Q}_{X_i}(2^{-n}) \text{ for } n \geqslant 1.$$

Also we define $J_{i,n} := [a_{i,n}, a_{i,n+1})$. If $a_{i,n} = a_{i,n+1}$ then $J_{i,n} = \emptyset$. We see that

$$2^{-n-1} < P(X_i > x_i) \leqslant 2^{-n}, x_i \in J_{i,n}, i = 1, \ldots, r.$$

In [2], it is proved that for $k = 1, \ldots, r$

$$\frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}|J_{k,n}|^{p_k} \leqslant E X_k^{p_k}$$

which implies

$$\left(\sum_{n=0}^{\infty} 2^{-n}|J_{k,n}|^{p_k}\right)^{1/p_k} \leqslant 2^{1/p_k}\|X_k\|_{p_k}. \tag{7}$$

We see that

$$\left\{J_{1,n_1} \times \cdots \times J_{r,n_r} : n_1, \ldots, n_r \in \{0, 1, 2, ..\}\right\}$$

is a collection of pairwise disjoint subsets of $[0, \infty)^r$ such that the support H is contained in the union of that collection. Therefore if we define for each $i_1, \ldots, i_r \in \{0, 1, 2, 3, \ldots\}$,

$$G_{i_1,\ldots,i_r} := \int \cdots \int_{(x_1,\ldots,x_r)\in J_{1,i_1}\times\cdots\times J_{r,i_r}} H(x_1, \ldots, x_r)\, dx_1 \cdots dx_r,$$

it is easy to see that

$$\int_0^{\infty} \ldots \int_0^{\infty} H(t_1, \ldots, t_r)\, dt_1 \ldots dt_r = \sum_{i_1=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1,\ldots,i_r}.$$

Since each of the $G_{i_1,\ldots,i_r}$'s is nonnegative, we have that

$$\begin{aligned}
&\sum_{i_1=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1,\ldots,i_r}\\
&\quad\leqslant \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1,i_1+i_2,\ldots,i_1+i_r} + \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1+i_2,i_1,i_1+i_3,\ldots,i_1+i_r}\\
&\qquad + \cdots + \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1+i_2,i_2+i_3,\ldots,i_1+i_r,i_1}.
\end{aligned} \tag{8}$$

We will find an upper bound for

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1,i_1+i_2,\ldots,i_1+i_r} = \sum_{i_2=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} \left(\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\ldots,i_1+i_r}\right). \tag{9}$$

The upper bounds for the other terms of (8) can be found in a similar way. In order to find the upper bound for (9), we will find two bounds for

$$\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}. \tag{10}$$

The first bound will be used when all the integers $i_2, \dots, i_r$ are "reasonably small" and the second one will be used when at least one of the integers $i_2, \dots, i_r$ is "large".

By (6) , for $(x_1, \dots, x_r) \in J_{1,i_1} \times \cdots \times J_{r,i_r}$ we have

$$H(x_1, \dots, x_r) \leqslant \min\{2^{-i_1}, \dots, 2^{-i_r}, \lambda \cdot 2^{-i_1/p_1-\cdots-i_r/p_r}\}$$

and hence

$$G_{i_1,\dots,i_r} \leqslant |J_{1,i_1}| \dots |J_{r,i_r}| \min\{2^{-i_1}, \dots, 2^{-i_r}, \lambda \cdot 2^{-i_1/p_1-\cdots-i_r/p_r}\}. \tag{11}$$

Let s be the number such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_r} + \frac{1}{s} = 1.$$

Obviously, $s > 1$. We defined s in order to apply Hölder's inequality. Choose M the nonnegative integer such that

$$M \leqslant s \cdot \log_2(1/\lambda) < M + 1.$$

For any nonnegative integers $i_2, \dots, i_r$,

$$\begin{aligned}
&\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r} \\
&\quad = \sum_{i_1=0}^{M-1} G_{i_1,i_1+i_2,\dots,i_1+i_r} + \sum_{i_1=M}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}.
\end{aligned} \tag{12}$$

First we will get a bound for the first term in the right-hand side of (12). Using (11), Hölder's inequality (with exponents $p_1, p_2, \dots, p_r$ and s), (7) and the definition of M we get

$$\begin{aligned}
&\sum_{i_1=0}^{M-1} G_{i_1,i_1+i_2,\dots,i_1+i_r} \\
&\quad \leqslant \sum_{i_1=0}^{M-1} \lambda \cdot 2^{-i_1/p_1} \cdot 2^{-(i_1+i_2)/p_2} \dots 2^{-(i_1+i_r)/p_r} \cdot |J_{1,i_1}| \cdot |J_{2,i_1+i_2}| \dots |J_{r,i_1+i_r}| \\
&\quad = \lambda \cdot \sum_{i_1=0}^{M-1} |J_{1,i_1}| \cdot 2^{-(i_1)/p_1} \left\{ \prod_{k=2}^{r} |J_{k,i_1+i_k}| \cdot 2^{-(i_1+i_k)/p_k} \right\} \cdot 1 \\
&\quad \leqslant \lambda \left(\sum_{i_1=0}^{M-1} |J_{1,i_1}|^{p_1} \cdot 2^{-i_1} \right)^{1/p_1}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left\{ \prod_{k=2}^{r} \left(\sum_{i_1=0}^{M-1} |J_{k,i_1+i_k}|^{p_k} \cdot 2^{-(i_1+i_k)} \right)^{1/p_k} \right\} \left(\sum_{m=0}^{M-1} 1^s \right)^{1/s} \\
&\leqslant \lambda \left(2 \cdot E|X_1|^{p_1}\right)^{1/p_1} \cdot \left\{ \prod_{k=2}^{r} \left(2 \cdot E|X_k|^{p_k}\right)^{1/p_k} \right\} M^{1/s} \\
&= \lambda \cdot 2^{1/p_1+\cdots+1/p_r} \cdot M^{1/s} \prod_{k=1}^{r} \|X_k\|_{p_k} \\
&\leqslant \lambda \cdot 2 \cdot (s \cdot \log_2(1/\lambda))^{1/s} \prod_{k=1}^{r} \|X_k\|_{p_k} \\
&\leqslant 4 \cdot \lambda \cdot (\log_2(1/\lambda))^{1/s} \prod_{k=1}^{r} \|X_k\|_{p_k}.
\end{aligned}
$$

For the last inequality, we used the fact that $s^{1/s} < 2$ which is easy to prove using calculus.

Now, we get an upper bound for the second term in the right-hand side of (12). Let $i_{\max} := \max\{i_2, \ldots, i_r\}$. Notice that $2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r} \leqslant 1$. Thus using (11), Hölder's inequality, (7) and the definition of M we have,

$$
\begin{aligned}
&\sum_{i_1=M}^{\infty} G_{i_1, i_1+i_2, \ldots, i_1+i_r} \\
&\leqslant \sum_{i_1=M}^{\infty} |J_{1,i_1}| \cdot |J_{2,i_1+i_2}| \ldots . |J_{r,i_1+i_r}| \cdot 2^{-i_1-i_{\max}} \\
&= 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r} \cdot \sum_{i_1=M}^{\infty} |J_{1,i_1}| \cdot 2^{-i_1/p_1} \left\{ \prod_{k=2}^{r} |J_{k,i_1+i_k}| \cdot 2^{-(i_1+i_k)/p_k} \right\} \cdot 2^{-i_1/s} \\
&\leqslant \sum_{i_1=M}^{\infty} |J_{1,i_1}| \cdot 2^{-i_1/p_1} \left\{ \prod_{k=2}^{r} |J_{k,i_1+i_k}| \cdot 2^{-(i_1+i_k)/p_k} \right\} \cdot 2^{-i_1/s} \\
&\leqslant \left(\sum_{i_1=M}^{\infty} \cdot |J_{1,i_1}|^{p_1} 2^{-i_1} \right)^{1/p_1} \left\{ \prod_{k=2}^{r} \left(\sum_{i_1=M}^{\infty} |J_{k,i_1+i_k}|^{p_k} \cdot 2^{-(i_1+i_k)} \right)^{1/p_k} \right\} \\
&\quad \times \left(\sum_{i_1=M}^{\infty} 2^{-i_i} \right)^{1/s} \\
&\leqslant 2^{1/p_1} \cdot \|X_1\|_{p_1} \left\{ \prod_{k=2}^{r} 2^{1/p_k} \|X_k\|_{p_k} \right\} (2^{-M+1})^{1/s} \\
&= 2^{1/p_1+\cdots+1/p_r} \cdot \left\{ \prod_{k=1}^{r} \|X_k\|_{p_k} \right\} \cdot (2^{-M+1})^{1/s} \\
&\leqslant 2^{1/p_1+\cdots+1/p_r} \cdot \left\{ \prod_{k=1}^{r} \|X_k\|_{p_k} \right\} \cdot (2^{-s\log_2(1/\lambda)+2})^{1/s}
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{1/p_1+\cdots+1/p_r} \cdot \left\{ \prod_{k=1}^{r} \|X_k\|_{p_k} \right\} \cdot \lambda \cdot 2^{2/s} \\
&= 2 \cdot 2^{1/s} \cdot \lambda \prod_{k=1}^{r} \|X_k\|_{p_k} \\
&\leqslant 4 \cdot \lambda \cdot \prod_{k=1}^{r} \|X_k\|_{p_k}.
\end{aligned}
$$

So combining the last two inequalities in (12), we get that for any nonnegative integers $i_2, \ldots, i_r$,

$$
\sum_{i_1=0}^{\infty} G_{i_1, i_1+i_2, \ldots, i_1+i_r} \leqslant 4\lambda \cdot (1 + (\log_2(1/\lambda))^{1/s}) \prod_{k=1}^{r} \|X_k\|_{p_k}. \tag{13}
$$

Now we get another inequality for (10). We recall that we defined $i_{\max} = \max\{i_2, \ldots, i_r\}$. By (11), Hölder's inequality and (7)

$$
\begin{aligned}
&\sum_{i_1=0}^{\infty} G_{i_1, i_1+i_2, \ldots, i_1+i_r} \\
&\quad \leqslant \sum_{i_1=0}^{\infty} |J_{1,i_1}| \cdot |J_{2,i_1+i_2}| \ldots |J_{r,i_1+i_r}| \cdot 2^{-(i_1+i_{\max})} \\
&\quad = 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r} \sum_{i_1=0}^{\infty} |J_{1,i_1}| \cdot 2^{-i_1/p_1} \left\{ \prod_{k=2}^{r} |J_{k,i_1+i_k}| 2^{-(i_1+i_k)/p_k} \right\} \cdot 2^{-i_1/s} \\
&\quad \leqslant 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r} \\
&\qquad \times \left(\sum_{i_1=0}^{\infty} |J_{1,i_1}|^{p_1} 2^{-i_1} \right)^{1/p_1} \cdot \left\{ \prod_{k=2}^{r} \left(\sum_{i_1=0}^{\infty} |J_{k,i_1+i_k}|^{p_k} 2^{-(i_1+i_k)} \right)^{1/p_k} \right\} \cdot \left(\sum_{i_1=0}^{\infty} 2^{-i_1} \right)^{1/s} \\
&\quad \leqslant 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r} \cdot 2^{1/p_1} \|X_1\|_{p_1} \ldots 2^{1/p_r} \|X_r\|_{p_r} \cdot 2^{1/s} \\
&\quad = 2 \cdot 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}.
\end{aligned} \tag{14}
$$

We are now ready to find an upper bound for (9). Let L be the nonnegative integer such that

$$
L - 1 \leqslant \max \left\{ \frac{p_1 \cdot s}{s + p_1}, \ldots, \frac{p_r \cdot s}{s + p_r} \right\} \cdot \log_2 \left(\frac{1}{\lambda} \right) < L.
$$

Then

$$
\begin{aligned}
&\sum_{i_1=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} G_{i_1, i_1+i_2, \ldots, i_1+i_r} \\
&\quad = \sum_{i_2=1}^{L} \cdots \sum_{i_r=1}^{L} \sum_{i_1=1}^{\infty} G_{i_1, i_1+i_2, \ldots, i_1+i_r} + \sum_{i_{\max} \geqslant L} \sum_{i_1=0}^{\infty} G_{i_1, i_1+i_2, \ldots, i_1+i_r}
\end{aligned} \tag{15}
$$

(here $\sum_{i_{\max} \geqslant L}$ means the sum over all $(i_2, \ldots, i_r) \in \{0, 1, 2, \ldots\}^{r-1}$ such that $i_{\max} \geqslant L$).

We will later bound the first term in the right-hand side of (15) using (13). Now we turn our attention to the second term. Since by (14) we have

$$\sum_{i_{\max}\geqslant L}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\ldots,i_1+i_r} \leqslant 2\cdot\prod_{i=1}^{r}\|X_i\|_{p_i}\cdot\sum_{i_{\max}\geqslant L} 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r},$$

we will get an upper bound for that last sum. For that purpose, note that

$$\begin{aligned}
&\sum_{i_{\max}\geqslant L} 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r}\\
&=\sum_A\left(\sum_{i_{\sigma(2)}=0}^{L-1}\cdots\sum_{i_{\sigma(a)}=0}^{L-1}\sum_{i_{\sigma(a+1)}=L}^{\infty}\cdots\sum_{i_{\sigma(r)}=L}^{\infty}2^{-i_{\max}+i_{\sigma(2)}/p_{\sigma(2)}+\cdots+i_{\sigma(r)}/p_{\sigma(r)}}\right)\\
&=\sum_A\left(\sum_{i_{\sigma(2)}=0}^{L-1}\cdots\sum_{i_{\sigma(a)}=0}^{L-1}2^{i_{\sigma(2)}/p_{\sigma(2)}+\cdots+i_{\sigma(a)}/p_{\sigma(a)}}\right)\\
&\quad\times\left(\sum_{i_{\sigma(a+1)}=L}^{\infty}\cdots\sum_{i_{\sigma(r)}=L}^{\infty}2^{-i_{\max}+i_{\sigma(a+1)}/p_{\sigma(a+1)}+\cdots+i_{\sigma(r)}/p_{\sigma(r)}}\right),
\end{aligned}\tag{16}$$

where each sum is taken over all subsets $A\subsetneq\{2,3,\ldots,r\}$ and for each A, we choose one permutation $\sigma:\{2,3,\ldots,r\}\to\{2,3,\ldots,r\}$ such that $\sigma(2),\ldots,\sigma(a)\in A$ and $\sigma(a+1),\ldots,\sigma(r)\notin A$ (where $a=1+Card(A)$). Using lemma 3 we get that

$$\begin{aligned}
&\sum_{i_{\sigma(a+1)}=L}^{\infty}\cdots\sum_{i_{\sigma(r)}=L}^{\infty}2^{-i_{\max}+i_{\sigma(a+1)}/p_{\sigma(a+1)}+\cdots+i_{\sigma(r)}/p_{\sigma(r)}}\\
&\leqslant\sum_{i_{\sigma(a+1)}=L}^{\infty}\cdots\sum_{i_{\sigma(r)}=L}^{\infty}2^{-i_{\max}\cdot(1/p_1+1/p_{\sigma(2)}+\cdots+1/p_{\sigma(a)}+1/s)}\\
&\leqslant(r-a)!\cdot\frac{2^{-L\cdot(1/p_1+1/p_{\sigma(2)}+\cdots+1/p_{\sigma(a)}+1/s)}}{(1-2^{-(1/p_1+1/s)})^{r-a}}
\end{aligned}\tag{17}$$

and by a direct computation using the geometric series formula

$$\begin{aligned}
&\sum_{i_{\sigma(2)}=0}^{L-1}\cdots\sum_{i_{\sigma(a)}=0}^{L-1}2^{i_{\sigma(2)}/p_{\sigma(2)}+\cdots+i_{\sigma(a)}/p_{\sigma(a)}}\\
&=\frac{2^{L/p_{\sigma(2)}}-1}{2^{1/p_{\sigma(2)}}-1}\cdots\frac{2^{L/p_{\sigma(a)}}-1}{2^{1/p_{\sigma(a)}}-1}\\
&\leqslant\frac{2^{L/p_{\sigma(2)}}}{2^{1/p_{\sigma(2)}}-1}\cdots\frac{2^{L/p_{\sigma(a)}}}{2^{1/p_{\sigma(a)}}-1}\\
&=\frac{2^{L\cdot(1/p_{\sigma(2)}+\cdots+1/p_{\sigma(a)})}}{(2^{1/p_{\sigma(2)}}-1)\cdots(2^{1/p_{\sigma(a)}}-1)}\\
&\leqslant 2^{L\cdot(1/p_{\sigma(2)}+\cdots+1/p_{\sigma(a)})}\left(\prod_{i=1}^{r}\frac{1}{2^{1/p_i}-1}\right).
\end{aligned}\tag{18}$$

So using (17) and (18) into (16)

$$
\begin{aligned}
&\sum_{i_{\max}\geqslant L}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}\\
&\quad\leqslant 2\cdot\prod_{i=1}^{r}\|X_i\|_{p_i}\sum_{i_{\max}\geqslant L} 2^{-i_{\max}+i_2/p_2+\cdots+i_r/p_r}\\
&\quad\leqslant 2\cdot\prod_{i=1}^{r}\|X_i\|_{p_i}\sum_{A}(r-a)!\cdot\frac{2^{-L\cdot(1/p_1+1/s)}}{\left(1-2^{-(1/p_1+1/s)}\right)^{r-a}}\prod_{i=1}^{r}\frac{1}{2^{1/p_i}-1}\\
&\quad= 2\cdot\prod_{i=1}^{r}\|X_i\|_{p_i}2^{-L\cdot(1/p_1+1/s)}\left(\sum_{A}(r-a)!\cdot\frac{1}{\left(1-2^{-(1/p_1+1/s)}\right)^{r-a}}\prod_{i=1}^{r}\frac{1}{2^{1/p_i}-1}\right)\\
&\quad= K_1\cdot 2^{-L\cdot(1/p_1+1/s)}\cdot\prod_{i=1}^{r}\|X_i\|_{p_i},
\end{aligned}\tag{19}
$$

where K_1 was defined by Eq. (5). Using (13), (19) and the definition of L

$$
\begin{aligned}
&\sum_{i_i=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}\\
&\quad=\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty}\left(\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}\right)\\
&\quad=\sum_{i_2=0}^{L-1}\cdots\sum_{i_r=0}^{L-1}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}+\sum_{i_{\max}\geqslant L}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}\\
&\quad\leqslant\sum_{i_2=0}^{L-1}\cdots\sum_{i_r=0}^{L-1}4\lambda\cdot(1+(\log_2(1/\lambda))^{1/s})\prod_{k=1}^{r}\|X_k\|_{p_k}+\sum_{i_{\max}\geqslant L}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}\\
&\quad=4\lambda\cdot L^{r-1}\cdot(1+(\log_2(1/\lambda))^{1/s})\prod_{k=1}^{r}\|X_k\|_{p_k}+\sum_{i_{\max}\geqslant L}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}\\
&\quad\leqslant 4\lambda\cdot L^{r-1}\cdot(1+(\log_2(1/\lambda))^{1/s})\prod_{k=1}^{r}\|X_k\|_{p_k}+K_1\cdot 2^{-L\cdot\left(\frac{1}{p_1}+\frac{1}{s}\right)}\prod_{i=1}^{r}\|X_i\|_{p_i}\\
&\quad\leqslant 4\lambda\left(\max\left\{\frac{p_1\cdot s}{s+p_1},\dots,\frac{p_r\cdot s}{s+p_r}\right\}\log_2(1/\lambda)+1\right)^{r-1}(1+(\log_2(1/\lambda))^{1/s})\prod_{k=1}^{r}\|X_k\|_{p_k}\\
&\qquad+K_1\cdot\lambda\cdot\prod_{i=1}^{r}\|X_i\|_{p_i}.\\
&\quad\leqslant 4\lambda[\max\{p_1,p_2,\dots,p_r\}]^{r-1}(1+\log_2(1/\lambda))^{r-1}(1+(\log_2(1/\lambda))^{1/s})\prod_{k=1}^{r}\|X_k\|_{p_k}\\
&\qquad+K_1\cdot\lambda\cdot\prod_{i=1}^{r}\|X_i\|_{p_i}.
\end{aligned}
$$

For the next to last inequality, we used the following facts that follow from the definition of L:

$$-L < -\log_2(1/\lambda)\frac{p_1 \cdot s}{s + p_1}$$

and

$$L \leqslant \max\left\{\frac{p_1 \cdot s}{s+p_1}, \dots, \frac{p_r \cdot s}{s+p_r}\right\} \cdot \log_2(1/\lambda) + 1.$$

By using the fact that $1 \leqslant (1 + \log_2(1/\lambda))^{r-1+1/s}$, proving $1 + (\log_2(1/\lambda))^{1/s} \leqslant 2^{1-1/s} \cdot (1 + \log_2(1/\lambda))^{1/s}$ using Hölder's inequality in two dimensions and other elementary facts, we see that

$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1, i_1+i_2, i_1+i_3, \dots, i_1+i_r} \leqslant C_1 \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}.$$

Similarly we can prove that

$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1+i_2, i_1, i_1+i_3, \dots, i_1+i_r} \leqslant C_2 \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}$$
$$\cdots$$
$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1+i_2, \dots, i_1+i_r, i_1} \leqslant C_r \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}.$$

Thus,

$$\begin{aligned}
&\int_0^\infty \dots \int_0^\infty H(t_1, \dots, t_r)\, dt_1 \dots dt_r \\
&\quad = \sum_{i_1=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1,\dots,i_r} \\
&\quad \leqslant \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1, i_1+i_2, \dots, i_1+i_r} + \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1+i_2, i_1, i_1+i_3, \dots, i_1+i_r} \\
&\qquad + \dots + \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty} G_{i_1+i_2, i_2+i_3, \dots, i_1+i_r, i_1} \\
&\quad \leqslant C_1 \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i} \\
&\qquad + C_2 \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i} \\
&\qquad \vdots \qquad \vdots
\end{aligned}$$

$$+C_r \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}$$

$$= C \cdot \lambda(1+\log(1/\lambda))^{r-1+1/s} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}$$

since $C = C_1 + \cdots + C_r$. This completes the proof of the Case 1.

Case 2: $\mathbf{p} \in (1,\infty)^r$ and $\sum_{i=1}^{n} p_i^{-1} = 1$.

Define $C = C(\mathbf{p})$ as in Case 1. We can choose a sequence $\{\mathbf{p_n} = (p_{1,n}, \ldots, p_{r,n})\}$ in $(1,\infty)^r$ such that $\sum_{i=1}^{r} p_{i,n}^{-1} < 1$ and $\mathbf{p_n} \to \mathbf{p}$. Thus by Case 1, for each $n \in \{1, 2, \ldots\}$

$$\int_0^\infty \cdots \int_0^\infty \min\left\{P(X_1 > t_1), \ldots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_{i,n}}\right\} dt_1 \ldots dt_r$$

$$\leqslant C(\mathbf{p_n}) \cdot \lambda(1-\log(\lambda))^{c(\mathbf{p_n})} \cdot \prod_{i=1}^{r} \|X_i\|_{p_{i,n}}. \tag{20}$$

We leave to the reader to check that as $n \to \infty$, $C(\mathbf{p_n}) \to C(\mathbf{p})$, $c(\mathbf{p_n}) \to c(\mathbf{p}) = r-1$ and that$\|X_i\|_{p_{i,n}} \to \|X_i\|_{p_i}$ for $i = 1, \ldots, r$ (note that the X_i's are simple). So by taking the limit $n \to \infty$ in (20) we get the result for Case 2.

Case 3: $\mathbf{p} \in (1,\infty]^r$ and $\sum_{i=1}^{n} p_i^{-1} \leqslant 1$.

Now we allow the possibility that some p_i's are equal to infinity. Without loss of generality, we can assume that for some $m \in \{0, \ldots, r\}$, $p_i < \infty$ for $i \leqslant m$ and $p_i = \infty$ for $i > m$. The proof will be written out here for the case $1 \leqslant m \leqslant r-1$. (If instead $m = 0$, the proof is quite trivial; and the case $m = r$ is covered in Cases 1 and 2.) We define $C = C(\mathbf{p}) := C(p_1, \ldots, p_m)$ from Case 1 and 2.

$$\int_0^\infty \cdots \int_0^\infty \min\left\{P(X_1 > t_1), \ldots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i}\right\} dt_1 \ldots dt_r$$

$$= \int_0^\infty \cdots \int_0^\infty \min\left\{P(X_1 > t_1), \ldots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{m} [P(X_i > t_i)]^{1/p_i}\right\} dt_1 \ldots dt_r$$

$$\leqslant \int_{t_1=0}^\infty \cdots \int_{t_m=0}^\infty \int_{t_{m+1}=0}^{\|X_{m+1}\|_\infty} \cdots \int_{t_r=0}^{\|X_r\|_\infty}$$

$$\times \min\left\{P(X_1 > t_1), \ldots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{m} [P(X_i > t_i)]^{1/p_i}\right\} dt_1 \ldots dt_r$$

$$\leqslant \int_{t_1=0}^\infty \cdots \int_{t_m=0}^\infty \int_{t_{m+1}=0}^{\|X_{m+1}\|_\infty} \cdots \int_{t_r=0}^{\|X_r\|_\infty}$$

$$\times \min\left\{P(X_1 > t_1), \ldots, P(X_k > t_m), \lambda \cdot \prod_{i=1}^{m} [P(X_i > t_i)]^{1/p_i}\right\} dt_1 \ldots dt_r$$

$$= \int_0^\infty \cdot\cdot \int_0^\infty \min\Big\{P(X_1 > t_1), \ldots, P(X_k > t_m),$$

$$
\left. \lambda \cdot \prod_{i=1}^{m} [P(X_i > t_i)]^{1/p_i} \right\} dt_1 \dots dt_m \prod_{i=m+1}^{r} \|X_i\|_\infty
$$

$$
\leqslant C \cdot \lambda (1 - \log(\lambda))^{m - \sum_{i=1}^{r} p_i^{-1}} \cdot \prod_{i=1}^{r} \|X_i\|_{p_i}.
$$

For the last inequality we applied the result obtained in Cases 1 and 2. This proves Case 3.

Case 4: $\mathbf{p} \in [1, \infty]^r$.

If $\mathbf{p} \in (1, \infty]$ then we are in one the previous cases. If one p_i's is equal to 1, then the condition $\sum_{i=1}^{r} p_i^{-1} = 1$ imply that all the other p_i's are equal to ∞. So we may assume without loss of generality that $p_1 = 1$ and $p_2 = \cdots = p_r = \infty$. We define $C = C(\mathbf{p}) = 1$. Then

$$
\int_0^\infty \cdots \int_0^\infty \min \left\{ P(X_1 > t_1), \dots, P(X_r > t_r), \lambda \cdot \prod_{i=1}^{r} [P(X_i > t_i)]^{1/p_i} \right\} dt_1 \dots dt_r
$$

$$
= \int_0^\infty \cdots \int_0^\infty \min \{ \lambda P(X_1 > t_1), \dots, P(X_r > t_r) \} \, dt_1 \dots dt_r
$$

$$
= \int_{t_1=0}^{\infty} \int_{t_2=0}^{\|X_2\|_\infty} \cdots \int_{t_r=0}^{\|X_r\|_\infty}
$$

$$
\times \min\{ \lambda P(X_1 > t_1), \dots, P(X_r > t_r) \} \, dt_1 \dots dt_r
$$

$$
\leqslant \int_0^\infty \lambda P(X_1 > t_1) \, dt_1 \prod_{i=2}^{r} \|X_i\|_\infty^r = \lambda \|X_1\|_1 \prod_{i=2}^{r} \|X_i\|_\infty^r.
$$

This proves Case 4 and thus Lemma 4 is proved. □

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References

[1] J. Bergh, J. Löfstöm, Interpolation Spaces, Springer, Berlin, 1976.

[2] R.C. Bradley, A covariance inequality under a two-part dependence assumption, Statist. Probab. Lett. 30 (1996) 287–293.

[3] R.C. Bradley, W. Bryc, Multilinear forms and measures of dependence between random variables, J. Multivariate Anal. 16 (1985) 335–367.

[4] W. Bryc, M. Peligrad, The central limit theorem for Tukey's $3R$ smoother, Statist. Probab. Lett. 13 (1992) 29–37.

[5] M. Peligrad, On the central limit theorem for weakly dependent sequences with a decomposed strong mixing coefficient, Stochastic Process. 42 (1992) 181–193.

[6] B.L.S. Prakasa Rao, Bounds for rth order joint cumulant under rth order strong mixing, Statist. Probab. Lett. 43 (1999) 427–431.

[7] E. Rio, Covariance inequalities for the strongly mixing processes, Ann. Inst. H. Poincaré Probab. Statist. 29 (1993) 587–597.

[8] E. Rio, Théorie Asymptotique des Processus Aléatoires Faiblement Dépendants, Mathématiques & Applications, vol. 31, Springer, Berlin, 2000.

[9] J.W. Tukey, Exploratory Data Analysis, Addison-Wesley, Reading, MA, 1977.